Fixed Point Properties in Homotopy Type Theory

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Contents

⚠ Our approach today will be informal, working from the topological instead of the type-theoretic viewpoint.

Fixed points in classical topology

Definition

A topological space X has the strict fixed point property (SFPP) if every continuous map $f : X \to X$ has a fixed point, i.e. a point $x \in X$ such that $f(x) = x$.

Brouwer's fixed point theorem: Bounded convex subspaces of $\mathbb{R}^n$ have the SFPP. Many more results like this in topology!

- These proofs are usually by contradiction, i.e., heavily rely on the law of excluded middle.
  No control over the *choice* of fixed point. In particular, it is not continuous with respect to the map.
  (exception: Banach's fixed point theorem, which uses metric spaces)
- Szymik [Szy15]: Fixed point properties where the choice of fixed point has to be continuous.
  However, no nontrivial finite-dimensional CW-complex satisfies this property.
- Not homotopy-invariant: $\{*\}$ has it, but $\mathbb{R}$ does not (e.g. $x \mapsto x + 1$).

We will consider a homotopy-invariant version of the SFPP that is also weaker than Szymik's in a model-independent setting [HoTT].

The language of HoTT/synthetic homotopy theory

- Idea: instead of building up topology from set theory, start with spaces and specify properties that we want them to have.
- Everything is a space, every map we can define is continuous. In particular: "for all x, $P(x)$" means that the choice of point witnessing $P(x)$ is continuous in x.
- Everything holds up to homotopy.
- We work completely constructively, no axiom of choice or law of excluded middle needed for the results in this talk.
- Results can be interpreted in any ∞-topos!

Start with a collection of basic spaces ($\varnothing$, $\{*\}$, $\mathbb{N}$, ...). For spaces X, Y, we can form

- (co)products $X \times Y$, $X \sqcup Y$;
- the mapping space $\mathrm{Map}(X, Y)$.

For $x, y \in X$, there is a path space $\mathrm{Path}_X(x, y)$. Always have constant paths $\mathrm{refl}_x \in \mathrm{Path}_X(x, x)$. Path spaces are subspaces of the free path space $\mathrm{Path}(X)$.

Inductively we have higher-dimensional cells between paths.

We form more complicated spaces, higher inductive types, by adding higher cells (compare with CW-complexes).

The language of HoTT/Synthetic homotopy theory

A homotopy between $f, g\colon X \to Y$ is a map $H\colon X \to \mathrm{Path}(Y)$ such that $H(x) \in \mathrm{Path}(f(x), g(x))$ for all $x \in X$. Write $H\colon f \sim g$.

We can define the space of homotopies $f \sim g$ and the space of equivalences $X \simeq Y$.

We assume we are working in a univalent universe of spaces $\mathcal{U}$, i.e. for all $X, Y \in \mathcal{U}$, $(X \simeq Y) \simeq \mathrm{Path}_{\mathcal{U}}(X, Y)$.

For any space X there is an n-truncation $\|X\|_n$ together with a map $\mathrm{tr}_n\colon X \to \|X\|_n$.

Universal property: $\mathrm{Map}(\|X\|_n, \|Y\|_n) \simeq \mathrm{Map}(X, \|Y\|_n)$.

A space X is a set if it is 0-truncated. We require groups to be sets.

The path component $X_{(x_0)}$ of $x_0 \in X$ is the space of all $x \in X$ such that $\|\mathrm{Path}(x_0, x)\|_{-1}$.
$\|-\|_{-1}$ here means we don't remember the choice of paths!

A space is a set iff its path components are contractible.

$\pi_n(X, x_0) := \|\Omega^n(X, x_0)\|_0$. Group structure given by concatenation of paths.

In particular, $\pi_0 X = \|X\|_0$.

Back to fixed points

Definition

A space X has the fixed point property (FPP) if there is a function $(s, H)\colon \operatorname{Map}(X, X) \to X \times \operatorname{Path}(X)$ such that $H(f) \in \operatorname{Path}(f(s(f)), s(f))$ for all $f : X \to X$.
In type theory:

$$\text{hasFixedPoints}(X) :\equiv \prod_{f:X\to X} \sum_{x\in X} \operatorname{Path}(f(x), x).$$

This is homotopy invariant! In particular, any contractible space works.

We can reproduce some of the proofs in [Szy15] in our setting:

Proposition (Szymik)

1. Every retract of a space with the FPP satisfies the FPP.
2. A product $X \times Y$ satisfies the FPP if and only if both X and Y satisfy the FPP. Not true for the SFPP!

Counterexamples

Definition

An H-space is a pointed space (X, x_0) equipped with an operation $\mu \colon X \times X \to X$ and homotopies $\mu(x_0, -) \sim \mathrm{const}_{x_0}$ and $\mu(-, x_0) \sim \mathrm{const}_{x_0}$.

Proposition

Let (X, μ) be an H-space such that for all $x \in X$, $\mu(-, x)$ is a homotopy equivalence. If X has the fixed point property, then it is contractible.

Therefore none of the following spaces have the fixed point property (unless they are contractible):

1. $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$*;
2. Eilenberg-Mac Lane spaces of Abelian groups;
3. loop spaces of any space;
4. Any products where one component is such a space, or any space that has such a space as a retract.

Maps between classifying spaces

The fundamental group functor π_1 induces an equivalence between the category of groups and the category of pointed, connected, 1-truncated spaces.

The inverse functor B sends a group G to its classifying space BG.

BG can be constructed as a higher inductive type [LF14].

The functorial action of B induces equivalences $\mathrm{Hom}(G, H) \simeq \mathrm{Map}_\bullet(BG, BH)$.What about $\mathrm{Map}(BG, BH)$?

Cagne-Buchholtz-Kraus-Bezem [CBKB], 2024: $\mathrm{Path}(BG, BG)_{(\mathrm{refl}_{BG})} \simeq BZ(G)$.

Theorem (Gottlieb, Christensen-C.)

Let G and H be groups.

1. *There is an equivalence* $\pi_0(\mathrm{Map}(BG, BH)) \simeq \mathrm{Hom}(G, H)/\sim_{\text{conj.}}$.
2. *For any* $v \in \mathrm{Hom}(G, H)$*, there is an isomorphism*

$$\pi_1(\mathrm{Map}(BG, BH), Bv) \simeq_{\mathrm{grp}} C_H(v(G)),$$

where $C_H(-)$ *is the centraliser.*

Note: If X is pointed connected, then $\mathrm{Map}(X, BH) \simeq \mathrm{Map}(B(\pi_1 X), BH)$.

Maps between classifying spaces: analysing the connected components

Corollary

Let G and H be groups. The path component of the map $\mathrm{const}_{\mathrm{base}_H} \in \mathrm{Map}(BG, BH)$ is precisely the space of all constant maps from BG to BH.

For self-maps of a general space:

Proposition (Christensen-C.)

Let (X, x_0) be a pointed, connected space. Then the function $\mathrm{const}_{(-)} : X \to \mathrm{Map}(X, X)_{(\mathrm{const}_{x_0})}$ is an equivalence if and only if the space $\mathrm{Map}_\bullet(X, \Omega X)$ is contractible.

If G and H are finite, we can completely describe the components of $\mathrm{Map}(BG, BH)$ without the axiom of choice:

Proposition (Christensen-C.)

Let G and H be finite groups. Then we can choose a pointed map in every path component of $\mathrm{Map}(BG, BH)$.

More generally, works if G, H and $\mathrm{Hom}(G, H)$ are compact types (as defined by Escardó) and H has decidable paths.

Maps between classifying spaces: the finite simple case

Corollary

Let G be a non-Abelian finite simple group. Then every self-map of BG is either constant or a pointed equivalence.

Proof.

Let $f : BG \to BG$. There is some $v \in \mathrm{Hom}(G, G)$ such that $\|f \sim Bv\|_{-1}$. This v is either constant or an isomorphism by the assumptions on G.

If v is an equivalence, then $C_G(v(G)) \simeq \{*\}$, so the component of Bv is contractible and $f \sim Bv$.

If v is constant, then f is also constant. $\square$

Corollary

Let G be a non-Abelian finite simple group. Then BG satisfies the fixed point property.

Generate a larger class of examples using products and retracts.

Formalisation

Our results are formally verified using the Rocq proof assistant. Available at `github.com/thchatzidiamantis/HoTT-FixedPoints/tree/FixedPoints`.

Questions

- The space $\mathrm{Map}(BS_3, BS_3)$ has a path component equivalent to $B(\mathbb{Z}/2\mathbb{Z})$. We suspect that this component fails.
- We do not know if any sphere other than $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$ and $\mathbb{S}^7$ also fails to satisfy the FPP.
- Bauer [Bau17]: fixed points using Lawvere's fixed point theorem. Analogue in nonstandard models of HoTT?
- Homotopy type of $\mathrm{Map}(X, K(G, n))$ for $n \geq 1$ in HoTT? Done for $K(\mathbb{Z}/2\mathbb{Z}, n)$ by Ljungström and Wärn.

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