Formalizing Simplicial Type Theory

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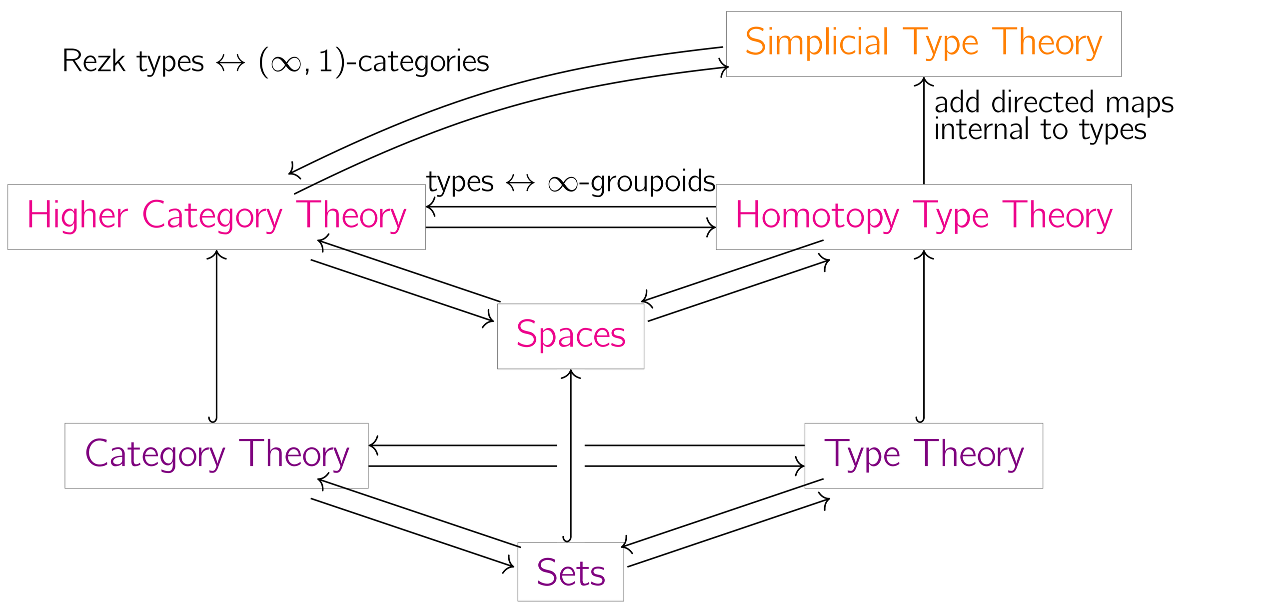
Acknowledgments. This work is part of an M.Sc. thesis project at the University of Bonn, supervised by Nima Rasekh. It consists of contributions to a collaborative formalization project in the Rzk proof assistant. We point to [3] for more details on Rzk and the ongoing project.

Why ∞-categories?

- Homotopy theory: A higher category with all objects, morphisms, 2-morphisms, etc., can replace spaces, continuous maps, homotopies, homotopies between homotopies... We obtain homotopy-coherent structures, which is not the case for $\mathcal{T}\mathbf{op}$.

  In the same way, a higher category with all its morphisms being invertible can replace the notion of a space, with its points, paths, homotopies and higher homotopies. Such objects are called ∞-groupoids.
- Category theory, higher algebra: Properties up to higher coherences, e.g. Associativity (A_∞-algebras), commutativity (E_∞-rings). Unique compositions up to homotopy.

Sets vs Types vs Categories - The higher version



Why synthetic ∞-categories?

Simplicial type theory, introduced in [1], extends homotopy type theory by equipping types with hom-types, i.e., types of directed arrows, which, under certain conditions, make types formally represent ∞-categories. We can:

- Prove statements about higher categories, independent of model.
- Do higher category theory in a simpler framework: Models of ∞-categories are generally hard to work with.
- Formalize such statements in a proof assistant. Type theory automatically provides us with computer-checkable proofs via different (possibly interactive) proof assistants.

Spaces and ∞-groupoids

Definition. The *simplex category* Δ has objects $[n] := \{0, 1, \dots, n\}$ and morphisms order-preserving maps. A *simplcial set* is a functor $\Delta^{op} \to \mathcal{S}\mathrm{et}$. $\mathcal{S} := \mathrm{Fun}(\Delta^{op}, \mathcal{S}\mathrm{et})$. We denote $X_n := X(n)$ for $X \in \mathrm{Obj}(\mathcal{S})$. We define the *standard n-simplex* $\Delta^n := \mathrm{Hom}_\Delta(-, [n])$. By the Yoneda lemma, $\mathrm{Hom}_{\mathcal{S}}(\Delta^n, X) = X_n$.

- We can visualize Δ^0 as a point 0, Δ^1 as a directed interval $0 \to 1$, Δ^2 as a filled triangle $\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & \Downarrow & 2 \end{tikzcd}$, Δ^3 as a tetrahedron, etc.
- We can now make the analogy: X_0 is the set of points, X_1 is maps, X_2 is homotopies between maps.
- We define the *horn* Λ^n_i as the union of all faces of Δ^n except the i-th one.

Definition. A simplicial set X is a *Kan complex* if all horns in X lift to a simplex: $\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$

Kan complexes are a model for ∞-groupoids! in Kan complexes behave like paths:

Example. If we have maps $f, g \in X_1$, we can form a horn $\begin{tikzcd} & x_1 \arrow[dr, "f"] & \\ x_0 \arrow[ur, "g"] & & x_2 \end{tikzcd}$ which lifts to $\begin{tikzcd} & x_1 \arrow[dr, "f"] & \\ x_0 \arrow[ur, "g"] \arrow[rr, "h"'] & \Downarrow & x_2 \end{tikzcd}$ and we say h is a composite of f and g. Then, there is a 3-simplex witnessing the uniqueness of such a composition up to homotopy.

Similarly, lifting a horn $\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[rr, "id"'] & & y \end{tikzcd}$ amounts to finding a homotopy inverse to f (at least on one side).

Theorem (Quillen)**.** *Topological spaes and Kan complexes present the same homotopy theory.*

Building $(\infty, 1)$-categories as simplicial objects in ∞-groupoids

We now introduce higher categories where arrows are *directed*, resembling morphisms rather than paths.

Definition. A *Segal space* W is a (Reedy fibrant) simplicial object in Kan complexes such that for all $n \geq 2$, the Segal maps $W_n \to W_1 \times_{W_0} W_1 \times_{W_0} \cdots \times_{W_0} W_1$ are equivalences.

With W_1 now as the *space* of maps, we can compose two composable maps by lifting along $X_2 \xrightarrow{\simeq} X_1 \times_{X_0} X_1$. One can construct the space $W_{hoequiv}$ of equivalences as a subspace of W_1.

Definition (Rezk)**.** A Segal space W is *complete* if the degeneracy $s_0 : W_0 \to W_{hoequiv}$ is an equivalence.

The higher groupoidal structure of types in homotopy type theory

For any $a, b : A$, there is an *identity type*

$$a =_A b$$

where terms are *paths* $p : a = b$.
Then we can have higher paths

$$H : p =_{(a =_A b)} q$$

between paths, paths between them, etc.

Theorem. *On paths, there are*

- *Inverses* $x =_A y \to y =_A x$
- *Concatenations* $x =_A y \to y =_A x \to x =_A z$
- *Witnesses for associativity, etc.*

Theorem (Kapulkin-Lumsdaine/Voevodsky)**.** *There is a model of homotopy type theory in* $\mathcal{S}$*.*

Building types with directed arrows as "simplicial objects in types"

In type theory, there is the *unit type* $\mathbf{1}$. Maps $\mathbf{1} \to A$ represent terms $a : A$. Simplicial type theory introduces:

- A *directed interval* type $\mathbf{2}$ with two points $0, 1 : \mathbf{2}$ and the relation $x : \mathbf{2} \vdash 0 \leq x \leq 1$.
- *Simplices* and their *boundaries* defined "geometrically": $\Delta^n := \{\langle t_1, \dots, t_n \rangle : \mathbf{2}^n \mid t_n \leq \cdots \leq t_1\}$. For example (informally), $\Delta^1 = \mathbf{2}$, $\partial\Delta^1 = \{t : \mathbf{2} \mid 0 \equiv t \vee t \equiv 1\}$. maps out of Δ^1 are arrows with the image of $\partial\Delta^1$ specifying the endpoints. Similarly, Δ^2 defines triples of maps.
- *Extension types* of the form $\left\langle \{t : J \mid \psi\} \to A|^{\phi}_{\alpha} \right\rangle$ used to define our lifts: This can be read as "maps that restrict to α when ϕ".

Definition. For a type A and $a, b : A$, we define the type of *arrows* $\mathrm{hom}_A(x, y) := \left\langle \Delta^1 \to A|^{\partial\Delta^1}_{[x,y]} \right\rangle$.
A is a *Segal type* if the type $\sum_{h : \mathrm{hom}_A(x,z)} \left\langle \Delta^2 \to A|^{\partial\Delta^2}_{[x,y,z,f,g,h]} \right\rangle$ is contractible for all $x, y, z : A$, $f : \mathrm{hom}_A(x, y)$, $g : \mathrm{hom}_A(y, z)$.
For $f : \mathrm{hom}_A(x, y)$, we have a type $\mathrm{isiso}(f)$ of witnesses that it is an isomorphism.
A is a *Rezk type* if $\mathrm{idtoiso} : (x =_A y) \to (x \cong_A y)$, where $(x \cong_A y) := \sum_{f : \mathrm{hom}_A(x,y)} \mathrm{isiso}(f)$, is an equivalence.

Theorem (Riehl-Shulman)**.** *There is a model of simplcial type theory in Reedy fibrant bisimplicial spaces, where Segal types correspond to Segal spaces and Rezk types correspond to complete Segal spaces.*

Formalization in the Rzk proof assistant

The Rzk proof assistant was developed by Nikolai Kudasov in order to formalize simplicial type theory. We first present examples of things we talked about, now written in Rzk:

hom-types

```
```rzk
#def hom
  ( A : U)
  ( x y : A)
  : U
  :=
    ( t : \Delta^1)
  -> A [ t == 0 |-> x ,
         t == 1 |-> y]
```
```

```
```rzk
#def hom2
  ( A : U)
  ( x y z : A)
  ( f : hom A x y)
  ( g : hom A y z)
  ( h : hom A x z)
 : U
 :=
    ( ( t1 , t2) : \Delta^2)
  -> A [ t2 == 0 |-> f t1 ,
         t1 == 1 |-> g t2 ,
         t2 == t1 |-> h t2]
 ```
```

Here the square brackets are notation for the respective extension types and U is a generic universe of types. ":" comes before the specified type, and ":=" before the specified term.

Example: Comparing dependent arrows and arrows in $\sum$-types

Our current formalization efforts are concentrated around the behavior of *covariant type families*. These are defined around the *dependent hom-types*: Given a type family $C : A \to U$, $u : C(x)$, $v : C(y)$ and $f : \mathrm{hom}_A(x, y)$, consider

$$\mathrm{dhom}_{C(f)}(u, v) := \left\langle \prod_{t:\mathbf{2}} C(f(t))|^{\partial\Delta^1}_{[u,v]} \right\rangle$$

In the code example on the right, we define a map

$$\left(\sum_{F : \mathrm{hom}_{\sum_{a:A} C(a)}((x,u),(y,v))} (\mathrm{pr}_1(F) = f) \right) \to \mathrm{dhom}_{C(f)}(u, v)$$

to provide a comparison between dependent maps and maps in the $\sum$-type "over" f.

More on covariant families and future work

Covariant type families have the property that the $\sum$-type defined by a covariant family over a Segal base type is Segal. Our current goal is to formalize a version of composition for dependent arrows. Moreover, we contribute formal proofs to the ("regular") homotopy type theory library of Rzk.

```
```rzk
#def hom-total-type-to-dhom
  ( A : U)
  ( is-segal-A : is-segal A)
  ( x y : A)
  ( f : hom A x y)
  ( C : A -> U)
  ( is-covariant-C : is-covariant A C)
  ( u : C x)
  ( v : C y)
  : ( \sum ( k : hom (total-type A C) (x , u) (y , v))
    , ( ( \ t -> first (k t)) =_{hom A x y} f))
  -> dhom A x y f C u v
  := \ (k , p) ->
       \ t ->
         ( transport
           ( hom A x y)
           ( \ g -> dhom A x y g C u v)
           ( \ r -> first (k r)) f p
             ( \ s -> second (k s)) t)
```
```

References

1. Emily Riehl and Michael Shulman, *A type theory for synthetic ∞-categories.* Higher Structures 1.1, 2017 , pp. 147-224.
2. Charles Rezk, *A Model for the Homotopy Theory of Homotopy Theory.* Transactions of the American Mathematical Society, vol. 353, no. 3, 2001, pp. 973-1007.
3. Nikolai Kudasov, Emily Riehl, and Jonathan Weinberger. *Formalizing the ∞-Categorical Yoneda Lemma.* In Proceedings of the 13th ACM SIGPLAN International Conference on Certified Programs and Proofs. Association for Computing Machinery, New York, NY, USA, 2024, pp. 274-290.

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